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Note

Every 5-Coloured Map in the Plane Contains a Monochrome Unit

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The minimum number of colours, m, needed to colour all the points in the Euclidean plane such that no two points unit distance apart are the same colour is still an unsolved problem, although in its known that $4 \leqslant m \leqslant 7$ (see [1, 2]). Woodall [3] proves that an infinite planar map requires at least six colours, but it is still not known whether seven are necessary.

It is convenient to introduce the term *Monochrome unit* to refer to a pair of points in E^2 unit distance apart having the same colour. Woodall's result may then be stated in the terms of the following theorem.

THEOREM. *Every 5-coloured map in the plane contains a monochrome unit.*

His proof makes use of an assertion that any simply connected Jordan region [4] containing an arc of the unit circle with length L greater than or equal to $\frac{2}{3}\pi$ must contain a monochrome unit if a map is constructed in its interior and each domain of the map is coloured one of two colours. The precise case $L = \frac{2}{3}\pi$ is then used to prove that every 5-coloured planar map which contains a vertex of degree 3 must contain a monochrome unit. Unfortunately it is possible to construct a counter-example to the assertion when $L = \frac{2}{3}\pi$ as follows.

Let A be the closed annulus bounded by the circles $|\mathbf{x}| = 1$ and $|\mathbf{x}| = 1 - h$, where $0 < h < 1$, and let R be the closed subset of A subtended by the angle $\frac{2}{3}\pi$ at the origin. The interior R° of R is, according to Woodall's definition, an interior arc of positive thickness. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be the end-points and mid-point, respectively, of the segment of $|\mathbf{x}| = 1$ which bounds R. Let e be the arc of unit radius centre $\mathbf{a}$ which cuts $|\mathbf{x}| = 1$ at $\mathbf{c}$ and divides R° into two disjoint regions S and T, where $\mathbf{a}$ lies on the boundary of S. R may be 2-coloured as follows: colour $\mathbf{a}$ red; colour S and the remainder of its

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boundary, including e, blue; colour T and its boundary, excluding e, red. Clearly R contains no monochrome units, and so neither does its interior R°. ∎

The theorem is still correct, and by replacing the above assertion with the following lemma it is possible to construct a proof on broadly similar lines to that of Woodall.

DEFINITION. Let A be any closed, bounded doubly connected set in E^2 containing the unit circle. If the removal of any point in A renders A simply connected then such a point is called a *cut point* of A. If A has no cut points its interior A° is said to be a *unit annulus*. If A has a finite number of cut points (which must occur on the unit circle) then A° is said to be a *finitely disconnected unit annulus*.

LEMMA. *Let A° be a finitely disconnected unit annulus for which the unit circle contained in its closure, A, has at least one segment of length greater than $\pi/3$ containing no cut points of A. Then any 2-colouring of A° contains a monochrome unit.*

The proofs of the lemma and theorem need care and attention to fine detail, requiring about two hundred lines of argument altogether.

REFERENCES

1. P. ERDÖS, R. L. GRAHAM, P. MONTGOMERY, B. L. ROTHSCHILD, J. SPENCER, AND E. G. STRAUS, Euclidean Ramsey theorems 1, *J. Combinatorial Theory Ser. A* **14** (1973), 341–363.
2. V. KLEE, Some unsolved problems in plane geometry, *Math. Mag.* **52** (1979), 131–145.
3. D. R. WOODALL, Distance realized by sets covering the plane, *J. Combinatorial Theory Ser. A* **14**(1973), 187–200.
4. T. RADÓ, "Length and Area," Amer. Math. Soc. Colloquium Publications No. 30, Amer. Math. Soc., Providence, R. I., 1948.