COLOURING THE PLANE WITH NO MONOCHROME UNITS

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An investigation is made of 4 and 5-colourings of the plane showing in each case that certain configurations cannot occur if no two points unit distance apart are to be allowed the same colour. This leads to a proof of the proposition that every 5-coloured planar map contains two points of the same colour unit distance apart. An earlier attempt at a proof exists in the literature, but this is shown to be flawed.
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The minimum number of colours, m, needed to colour all the points in the Euclidean plane such that no two points unit distance apart are the same colour is still an unsolved problem, although it is known that $4 \leq m \leq 7$ (see [1] and [2]). Woodall[3] proves that an infinite planar map requires at least six colours, but it is still not known whether seven are necessary.

It is convenient to introduce the term MONOCHROME UNIT to refer to a pair of points in E^2 unit distance apart having the same colour. This paper investigates 4 and 5-colourings of E^2 and shows that if no monochrome units are to occur then certain elementary configurations must be excluded. These results are then used to prove that every 5-coloured map in the plane contains a monochrome unit, so confirming the result of Woodall[3].

Woodall's proof makes use of an assertion that any simply connected Jordan region[4] containing an arc of the unit circle with length L greater than or equal to $2\pi/3$ must contain a monochrome unit if a map is constructed in its interior and each domain of the map is coloured one of two colours. Unfortunately it is possible to construct a counter example for the case L= $2\pi/3$.

Let A be the closed annulus bounded by the circles $|\underline{x}|=1$ and $|\underline{x}|=1-h$, where $0<h<1$, and let R be the closed subset of A subtended by the angle $2\pi/3$ at the origin. The interior R^0 of R is, according to Woodall's definition, an interior arc of positive thickness. Let $\underline{a}$, $\underline{b}$ and $\underline{c}$ be the end-points and mid-point respectively of the segment of $|\underline{x}|=1$ which bounds R. Let e be the arc of unit radius centre $\underline{a}$ which cuts $|\underline{x}|=1$ at $\underline{c}$ and divides R^0 into two disjoint regions S and T, where $\underline{a}$ lies on the boundary of S. R may be 2-coloured as follows: colour $\underline{a}$ red; colour S and the remainder of its boundary, including e, blue; colour T and its boundary, excluding e red. Clearly R contains no monochrome units, and so neither does its interior R^0.$\square$

Woodall uses the case L= $2\pi/3$ to prove that every 5-coloured planar map which contains a vertex of degree 3 must contain a monochrome unit. In the light of the above counter example his theorem, although correct, requires a more careful proof.

In order to proceed further the following definitions are required.

DEFINITION 1
Let S and T be subsets of E^2. S is said to <u>subtend T at unit distance</u> if T is the union of all unit circles centred on points in S.

DEFINITION 2
Let A be any closed, bounded doubly connected set in E^2 containing the unit circle. If the removal of any point in A renders A simply connected then such a point is called a cut point of A. If A has no cut points its interior A^0 is said to be a unit annulus. If A has a finite number of cut points (which must occur on the unit circle) then A^0 is said to be a finitely disconnected unit annulus.

DEFINITION 3
A planar map is an ordered pair M(S,B) where S is a set of mutually disjoint bounded finitely connected open sets (regions) in E^2 and B is a set of simple closed curves (frontiers) in E^2 satisfying
(i) the union of the members of S and B forms a covering of E^2;
(ii) $\exists$ a one-to-one function $F:S\rightarrow B$ such that $b = F(s)$, $s\in S$, is the exterior boundary of s;
(iii) the boundary of $s\in S$ is the union of F(s) and at most a finite number of other members of B, which are the interior boundaries of s.

A point on the boundary of s is called a boundary point of s. A boundary point which lies on the boundary of k regions, $k\geq 3$, is called a vertex of degree k. A closed subset of a frontier $b\in B$ which is bounded by two vertices and contains no other vertices is called an edge of each region for which b is part of the boundary. Two regions are adjacent if their boundaries contain a common edge or a common frontier.

The above definition is more general than the usual definition of a planar map (see, for example, [3]) which requires each region $s\in S$ to be simply connected, and requires each frontier $b\in B$ to contain at least two vertices.

DEFINITION 4
An r-colouring of a planar map is a function $C_r:E^2\rightarrow\{c_1,c_2,\ldots,c_r\}$ where C_r is constant over each region in S and where a boundary

point is given the colour of one of the regions in the closure of which it lies.

To prove that an r-coloured map must contain a monochrome unit it is sufficient to examine only those r-coloured maps satisfying

(i) each region has no interior boundaries, i.e. its closure does not contain the closure of any other region;

(ii) different regions of the same colour have no common boundary points.

This is best understood by observing that every r-coloured map with no monochrome units may be simplified to an r-coloured map with no monochrome units satisfying (i) and (ii) above as follows.

(a) For each region s with interior boundaries, remove these boundaries and assimilate into s all regions whose closures are contained in the closure of s.

(b) Remove any edges common to adjacent regions of the same colour.

(c) For each vertex $\underline{v}$ which is a boundary point of two non-adjacent regions of the same colour, choose $\varepsilon>0$ sufficiently small and describe an ε-neighbourhood whose closure contains $\underline{v}$ and whose intersection with each of the two regions is non-null, colouring this ε-neighbourhood the same colour as the two regions, and thus forming one new region incorporating the original two and the ε-neighbourhood.

A sequence of theorems now follows, concluding with the main result of this paper that every 5-coloured planar map contains a monochrome unit.

THEOREM 1

Let A^0 be a finitely disconnected unit annulus (see Definition 2) for which the unit circle contained in its closure, A, has at least one segment of length greater than $\pi/3$ containing no cut points of A. Then any 2-colouring of A^0 contains a monochrome unit.

LEMMA

Let γ be any simple arc[4] of length L containing at least two points unit distance apart. If γ is 2-coloured with no monochrome units then given ε>0 ∃ an ε-neighbourhood in γ containing a point of each colour.

Proof

There exist two points $\underline{x}_1$ and $\underline{y}_1$ in γ, not both the same colour, with $\left|\underline{x}_1 - \underline{y}_1\right| = 1$. Let $\varepsilon > 0$ be given. The following algorithm uses the method of bisection[5] to prove the lemma.

1. Set i=1.
2. Let $\underline{w}_i$ be the point in γ mid-way (by arc-length) between $\underline{x}_i$ and $\underline{y}_i$.
3. If the colours of $\underline{w}_i$ and $\underline{x}_i$ are not the same then put $\underline{x}_{i+1} = \underline{x}_i$ and $\underline{y}_{i+1} = \underline{w}_i$ otherwise put $\underline{x}_{i+1} = \underline{w}_i$ and $\underline{y}_{i+1} = \underline{y}_i$.
4. If $\left|\underline{x}_{i+1} - \underline{y}_{i+1}\right| \geq \varepsilon$ increase i by 1 and re-cycle from 2, otherwise stop.

The algorithm terminates in not more than n cycles, where $\varepsilon 2^n > L$. □

Proof (Theorem 1)

Suppose A can be 2-coloured with no monochrome units. There exists an infinite family Γ of simple closed curves intersecting one another only at the cut points of A, each, apart from the cut points of A, lying entirely within A^0, and for each of which there is a segment of finite length containing two points unit distance apart not separated by a cut point. This segment contains, for any given ε>0, an ε-neighbourhood in which lies a point of each colour (by the lemma). Let $\gamma_1 \in \Gamma$ be such that every point on γ_1 in A^0 is unit distance from at most one of the cut points of A (clearly only a

finite number of points are unit distance from two or more of the cut points of A, and γ_1 may be chosen to avoid all those that lie in A^0).
Now we can find $\delta\in(0,1)$ such that for every $\varepsilon\in(0,\delta)$ $\exists$ an ε-neighbourhood on γ_1 in A^0 containing a point of each colour and containing at most one point which is unit distance from a cut point of A. Let $\underline{x}$ and $\underline{y}$ be points of each colour in such an ε-neighbourhood, and suppose $\underline{x}$ is unit distance from a cut point, $\underline{c}$, of A.

Let $\gamma_2 \in \Gamma$. $\exists$ an arc α in A^0 of unit radius and centre $\underline{x}$ which intersects γ_1 at $\underline{x}'$ and γ_2 at $\underline{x}''$ (neither of which is a cut point of A), and $\exists$ an arc β in A^0 of unit radius and centre $\underline{y}$ which intersects γ_1 at $\underline{y}'$ and γ_2 at $\underline{y}''$ (again neither of which is a cut point of A). α and β are chosen such that $\underline{x}'$ and $\underline{y}'$ are further from $\underline{c}$ than from $\underline{x}$ and $\underline{y}$ respectively (by arc-length along γ_1).

Let P and Q be sets subtended at unit distance by α and β respectively. P and Q are disconnected annuli, each having one cut point at $\underline{x}$ and $\underline{y}$ respectively, and each intersecting A^0 in a band of finite width between γ_1 and γ_2. Let these bands be respectively P' and Q'. Q' may be considered to be the image of P' under a homeomorphism T which depends on $|\underline{x}-\underline{y}|$. Defining $d(P',Q') = \sup\{|p-T(p)|:p\in P'\}$ we have $d(P',Q')\to 0$ as $|x-y|\to 0$; in this sense we say $P'\to Q'$ as $|x-y|\to 0$. There must then exist $\varepsilon>0$ such that for $|\underline{x}-\underline{y}|<\varepsilon$, $P'\cap Q'\neq 0$. But $P'\cap Q'$ must be coloured differently to both $\underline{x}$ and $\underline{y}$, and so A^0 must be 3-coloured at least.□

Using this result it is possible to exclude two configurations from any 4-colouring of E^2 without monochrome units, and show as a natural consequence that any 4-coloured map in E^2 contains a monochrome unit.

THEOREM 2

Let E^2 be 4-coloured. If for some distinct $\underline{x}$ and $\underline{y}$ $\exists$ two simple arcs with endpoints $\underline{x}$ and $\underline{y}$ each, excepting the endpoints, being monochrome but not both the same colour, then E^2 contains a monochrome unit.

Proof

Let the two simple arcs be γ and δ. If $|\underline{x}-\underline{y}|>1$ then both γ and δ contain a monochrome unit. If $|\underline{x}-\underline{y}|\leq 1$then the intersection of the sets subtended at unit distance by γ and δ (excluding the endpoints) is a disconnected annulus with at most two cut points. This annulus is 2-coloured, and so by Theorem 1 contains $\square$a monochrome unit.

THEOREM 3

If a 4-colouring of E^2 contains two differently coloured, bounded, open connected monochrome sets with a common segment of boundary of finite length, then E^2 contains a monochrome unit.

Proof

Let E and F be two such sets, and let $\underline{x}$ and $\underline{y}$ be two distinct points on the common segment of boundary. Because the closure of E is a simply connected Jordan region, $\exists$ a simple arc γ with endpoints $\underline{x}$ and $\underline{y}$ which, apart from its endpoints, lies in E[4]. There exists a similar arc δ in F. By Theorem 2 E^2 contains a monochrome unit.$\square$

Corollary

Every 4-coloured planar map contains a monochrome unit.

A similar result involving three sets can be proved for 5-colourings of E^2, and again the consequence is that every 5-coloured planar map contains a monochrome unit, but this requires careful proof.

THEOREM 4

If a 5-colouring of E^2 contains three disjoint, differently coloured, bounded, open, connected, monochrome sets each having two or more common boundary points with each of the other two, and all three having one common boundary point, then E^2 contains a monochrome unit.

Proof

Let $\underline{v}$ be the boundary point common to all three sets and let $\underline{a}_1$, $\underline{a}_2$ and $\underline{a}_3$ respectively be boundary points common to each pair of sets. We assume these points are distinct and not more than one unit from each other. $\exists$ simple closed curves γ_1 coloured c_1 containing $\underline{v}$, $\underline{a}_1$ and $\underline{a}_2$, γ_2 coloured c_2 containing $\underline{v}$, $\underline{a}_1$ and $\underline{a}_3$, and γ_3 coloured c_3 containing $\underline{v}$, $\underline{a}_2$ and $\underline{a}_3$, where in each case the colouring refers to every point on the curve with the possible exception of the points $\underline{v}$, $\underline{a}_1$, $\underline{a}_2$ and $\underline{a}_3$. Let P be the intersection of the sets subtended at unit distance by γ_1, γ_2 and γ_3, excepting the points $\underline{v}$, $\underline{a}_1$, $\underline{a}_2$ and $\underline{a}_3$. P is either a unit annulus or a finitely disconnected unit annulus with at most three cut points. P satisfies the requirements of Theorem 1, and since it is 2-coloured (viz. not c_1, c_2 or c_3) it must contain a monochrome unit.□

Corollary

Every 5-coloured planar map containing a vertex of degree 3 contains a monochrome unit.

THEOREM 5

Every 5-coloured planar map contains a monochrome unit.

Proof

We show (i) that every 5-coloured planar map with no monochrome units contains a vertex of degree 3 or 4 and (ii) that every such map containing a vertex of degree 4 also contains a vertex of degree 3.

(i) Let $\underline{v}$ be any vertex in a 5-coloured planar map with no monochrome units. Let γ be the boundary of one of the regions which has $\underline{v}$ as a boundary point, and let $\underline{a}$ and $\underline{b}$ be two other points on γ. There is a simple closed curve γ_1

passing through $\underline{v}$, $\underline{a}$, and $\underline{b}$ all the points of which, except possibly $\underline{v}$, $\underline{a}$, and $\underline{b}$, are coloured c_1. There is a simple closed curve γ_2 passing through $\underline{a}$, and $\underline{v}$ all the points of which, except possibly $\underline{a}$, and $\underline{v}$, are coloured c_2, and there is a simple closed curve γ_3 passing through $\underline{b}$, and $\underline{v}$ all the points of which, except possibly $\underline{b}$, and $\underline{v}$, are coloured c_3. Let T_2 be the intersection of the sets subtended at unit distance by γ_1 and γ_2 and let T_3 be the intersection of the sets subtended at unit distance by γ_1 and γ_3. The interiors of T_2 and T_3, T_2^0 and T_3^0 respectively, are unit annuli each rendered simply connected by at most one cut point, and so by Theorem 1 cannot be 2-coloured. T_2^0 must contain regions coloured c_3, c_4 and c_5, and T_3^0 must contain regions coloured c_2, c_4 and c_5. The interior of $T_1 = T_2 \cup T_3$ is a 4-coloured unit annulus. There is a vertex in T_1^0 which must be of degree 3 or 4. If not then there must be edges in T_1 which cut T_1 (rendering it simply connected) without intersecting any other edges. This is only possible if these edges separate regions coloured c_4 and c_5, except possibly those edges which contain the cut points of T_2 and T_3 (if these exist). But then T_1 contains a 2-coloured unit annulus with no more than two cut points, and must therefore by Theorem 1 contain a monochrome unit.

(ii) Let $\underline{v}$ be a vertex of degree 4 - if none exists then the proof is completed. Let $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{d}$ be points on the four edges incident to $\underline{v}$, and let c_1, c_2, c_3 and c_4 be the colours of the four regions of which $\underline{v}$ is a boundary point. There are four simple closed curves γ_1, γ_2, γ_3 and γ_4, each of which contains $\underline{v}$ and exactly two of {$\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}$}, the points on each curve being coloured respectively c_1, c_2, c_3 and c_4 except possibly the points $\underline{v}$, $\underline{a}$, $\underline{b}$, $\underline{c}$ and $\underline{d}$. Let the order of the γ_i be chosen such that γ_2 and γ_4 have only the point $\underline{v}$ in common.

Let T_i, i=1..4, be the intersection of sets subtended at unit distance by γ_j, j=1..4, $j \neq i$, and let $T = \cup T_i$. The interior of T, T^0, is a unit annulus, centre v, and every boundary point of a region in T^0 is a boundary point of at most three regions. Suppose none of these boundary points is a vertex. Then there exist edges which cut T (rendering it simply connected), some of which cut either both of T_1 and T_3 or both of T_2 and T_4. It is possible for an edge to cut T and only cut one of T_1 and T_3 or one of T_2 and T_4, but such an edge must intersect the unit circle centre v at one of at most four points, these points being the cut points of the finitely disconnected annuli which are the interiors of $T_1 \cup T_2$ and $T_3 \cup T_4$. There must be edges cutting T which intersect the unit circle centre v at points other than these four. This implies there are two regions with different colours each of which has points in both T_1 and T_3 or both T_2 and T_4. But the only colour common to both T_1 and T_3 or both T_2 and T_4 is c_5. So we arrive at a contradiction. Hence there must be vertices in T^0, and these are of degree 3.

By theorem 4 every 5-coloured map contains a monochrome unit.□

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