24
Chromatic Number of the Plane Meets Map Coloring: Townsend–Woodall's 5-Color Theorem

In Chapter 8, I described Douglas R. Woodall's attempt to obtain a result on the chromatic number of the plane under an additional condition that monochromatic sets are closed or simultaneously divisible into regions [Woo1]. Six years after his publication, Stephen P. Townsend found a logical mistake in Woodall's proof, constructed a counterexample showing that Woodall's proof cannot work, and went on to discover his own proof of the following major result.

Townsend–Woodall's 5-Color Theorem 24.1 [Tow2]. Every 5-colored planar map contains two points of the same color until distance apart.

This implies result 8.1:

Townsend–Woodall's Theorem 24.1' The chromatic number of the plane under map-type coloring is 6 or 7

In this chapter, I will give you the story of the proof and the proof itself.

24.1 On Stephen P. Townsend's 1979 Proof

This story must remind the readers of the famed Victorian Affair, which we discussed in Chapters 19 and 20. To sum it up, in 1879 Alfred B. Kempe published a proof of the 4-Color Map-Coloring Theorem, in which 11 years later Percy J. Heawood found an error and constructed a counterexample to demonstrate the irreparability of the hole. Heawood salvaged Kempe's proof as the 5-Color Theorem, but the 4CC had to wait nearly another century for its proof.

Our present story started with Douglas R. Woodall's 1973 publication, in which 6 years later Steven P. Townsend found an error and constructed a counterexample to demonstrate the irreparability of the hole. So far the two stories are so close! However, unlike its Victorian counterpart, Townsend went on to prove Woodall's statement, and so I thought the new story had a happy end—until February 11, 2007, when I asked Stephen about "the story of the proof." The surprising reply reached me by e-mail on February 20, 2007:

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> Story of the proof
>
> I first became interested in the plane-colouring problem in 1977 or 1978. At that time I was a lecturer in the Department of Mathematics at the University of Aberdeen, having just completed my doctoral thesis (in Numerical Analysis). I had read an article that listed some of the unsolved problems in Combinatorics at that time, and this one caught my attention.
>
> I was totally unaware of Douglas's 1973 proof, which was both my folly and my good fortune. Folly, in that I should have conducted a more exhaustive literature search before devoting time to the problem. Good fortune in that had I been aware of Douglas's paper I would not have spent any time on the problem; I certainly would not have had the temerity to check Douglas's proof for accuracy. It should be noted that I was a numerical analyst, not a combinatorialist, so my awareness of the field of combinatorics was somewhat limited, in spite of brushing shoulders at Aberdeen with some eminent contributors to the field.
>
> It was not until I had completed the proof, and was considering what references to include, that I came upon Douglas's paper. I was both devastated and puzzled. The puzzlement came from my intimate knowledge of the difficulties of certain aspects of the proof and the fact that Douglas seemed to have produced a proof that circumnavigated these difficulties. So it was with an attitude of "how did he manage this?" that I went through his proof and consequently spotted the error.
>
> A colleague at Aberdeen, John Sheehan, whom I'm sure you will have come across, encouraged me nonetheless to submit my proof for publication, but including a reference to Douglas's work. The rest I think you know.

Yes, Stephen Townsend was lucky, for not only was he the first to produce a proof—but he also rediscovered the statement of the result on his own, albeit after Woodall's publication—and this Townsend's rediscovery was a necessary condition for finding the proof.

However, Townsend's good luck, ran into a wall, when the *Journal of Combinatorial Theory*'s Managing Editor and the distinguished Ramsey theorist Bruce L. Rothschild wrote to Townsend on April 3, 1980:

> The Journal of Combinatorial Theory – Series A is now trying very hard to reduce its large backlog, and we ask all our referees to be especially attentive to the question of the importance of the papers. In this case the referee thought that the result was not of great importance. In view of our backlog situation then, we are reluctant to publish the paper. However, since it does correct an error in a previously published paper, we would like to have a very short note about it. Perhaps, you would be willing to do the following: Write a note pointing out the error, stating the theorem (Theorem 1) (without proof) used to get around the trouble, and that the theorem must be used with care to get around the problem.

Stephen P. Townsend had satisfied the Editor (what choice did he have!), and produced a 2-page proof-free note [Tow1], which was published the following year. This is where the story was to end in 1981.

No blame should be directed at Douglas R. Woodall, we all make mistakes (except those of us who do nothing). The mistake notwithstanding, Woodall's 1973 paper has remained one of the fine works on the subject. Moreover, he was the first to alert me of his mistake and Townsend's 2-page note. "I am a fan of your 1973 paper," I wrote to Woodall in the October 10, 1993 e-mail, in which I called [Tow1] "the Townsend's addendum." The following day Woodall replied as follows:

> I will put a reprint in the post to you today, together with a photocopy of Townsend's "addendum," as you so tactfully describe it. (The fact is, I boobed, and Townsend corrected my mistake.)

However, regret is in order about the decision by the *Journal of Combinatorial Theory Series A* (*JCTA*). While they apparently (and correctly) assessed Woodall's paper as being "of great importance" (an impossible test if one interprets it literally), they denied its readers—and the world—the pleasure and the profit of reading Townsend's proof of the major result.

I have corrected *JCTA*'s quarter-a-century old mistake, when I published Townsend's work in the April 2005 issue of *Geombinatorics* [Tow2]. Townsend's work was preceded by my historical introduction [Soi25], a version of which you have just read. I ended that introduction with the words I would like to repeat here: It gives me a great pleasure to introduce and publish Townsend's proof. In my opinion, it *is* of great importance—judge for yourselves!

It pains me to see that most researchers in the field are still unaware of Woodall's mistake and Townsend's proof. It suffices to look at the major problem books to notice that: not only the 1991 book by Croft–Falconer–Guy [CFG], but even the recent 2005 book by Brass–Moser–Pach [BMP] give credit to Woodall and do not mention Townsend! I hope this chapter will inform my colleagues of the correct credit and of Townsend's achievement.

Stephen Phillip Townsend was born on July 17, 1948 in Woolwich, London, England. He received both graduate degrees, Master's (1972) and doctorate (1977) from the University of Oxford. Townsend has been a faculty first in the department of mathematics (1974–1980) and then in the department of computer science (1982–present) at the University of Aberdeen, Scotland. Since 1995 he has also been Director of Studies (Admissions) in Sciences. In addition to publications in mathematics, Steven's list of publications includes "Women in the Church–Ordination or Subordination?" (1997).

24.2 Proof of Townsend–Woodall's 5-Color Theorem

In this chapter, I will present Stephen P. Townsend's proof. As you now know, it first appeared in 2005 in *Geombinatorics* [Tow2]. However, when I was preparing this chapter, I asked Stephen to improve the exposition, make his important proof more accessible to the reader not previously familiar with topology, and include plenty of drawings to help the reader to visualize the proof. He did it, quite brilliantly. Thus, presented below exposition of the proof has been written by Professor Townsend especially for this book in 2007.

He starts with a few basic definitions from general (point set) topology.

Definitions A pair of points in E^2 unit distance apart having the same color is called a *monochrome unit*.

Let S and T be subsets of E^2. S is said to *subtend T at unit distance* if T is the union of all unit circles centered on points in S.

Let A be any closed, bounded doubly connected set in E^2 containing a circle of unit radius. If the removal of any point in A renders it simply connected then such a point is called a *cut point* of A. If A has no cut points, its interior A^0 is said to be a *unit annulus*. If A has a finite number of cut points (which must occur on a circle of unit radius) then A^0 is said to be a *finitely disconnected unit annulus* (Fig. 24.1).

A *planar map* (Fig. 24.2) is an ordered pair M(S,B) where S is a set of mutually disjoint bounded finitely connected open sets (*regions*) in E^2 and B is a set of simple closed curves (*frontiers*) in E^2 satisfying

i. the union of the members of S and B forms a covering of E^2;
ii. there exists a one-to-one function $F : S \rightarrow B$ such that $b = F(s),\ s \in S$, is the *exterior boundary* of s;
iii. the *boundary* of $s \in S$ is the union of F(s) and at most a finite number of other members of B, which are the *interior boundaries* of s.

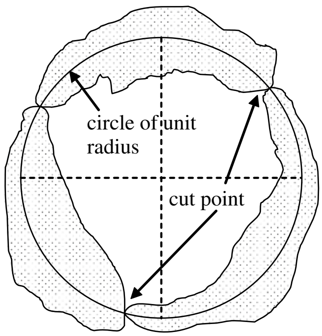

Fig. 24.1

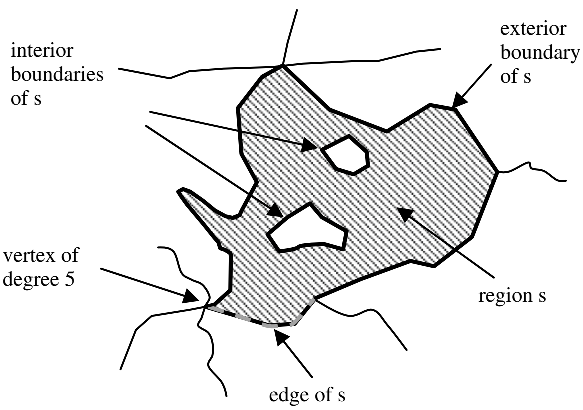

Fig. 24.2

A point on the boundary of s is called a *boundary point* of s. A boundary point, which lies on the boundary of k regions, $k \geq 3$, is called a *vertex of degree k*. A closed subset of a frontier $b \in B$, which is bounded by two vertices and contains no other vertices, is called an *edge* of each region for which b is part of the boundary. Two regions are *adjacent* if their boundaries contain a common edge or a common frontier.

The above definition is more general than the usual definition of a planar map, which requires each region $s \in S$ to be simply connected, and requires each frontier $b \in B$ to contain at least two vertices.

An *r-coloring* of a planar map is a function of $C_r : E^2 \to \{c_1, c_2, \ldots, c_r\}$ where C_r is constant over each region in S and where a boundary point is given the color of one of the regions in the closure of which it lies.

Initial Observations: To prove that an r-colored map must contain a monochrome unit it is sufficient to examine only those r-colored maps satisfying

(i) each region has no interior boundaries, i.e., its closure does not contain the closure of any other region;
(ii) different regions of the same color have no common boundary points.

This is best understood by observing that every r-colored map with no monochrome units may be simplified to an r-colored map with no monochrome units satisfying (i) and (ii) above as follows.

(a) For each region s with interior boundaries, remove these boundaries and assimilate into s all regions whose closures are contained in the closure of s.

(b) Remove any edges common to adjacent regions of the same color.

(c) For each vertex $\underline{v}$ which is a boundary point of two non-adjacent regions of the same color, choose $\varepsilon > 0$ sufficiently small and describe an ε-neighborhood whose closure contains $\underline{v}$ and whose intersection with each of the two regions is non-null, coloring this ε-neighborhood the same color as the two regions, and thus forming one new region incorporating the original two and the ε-neighborhood. (Fig. 24.3.)

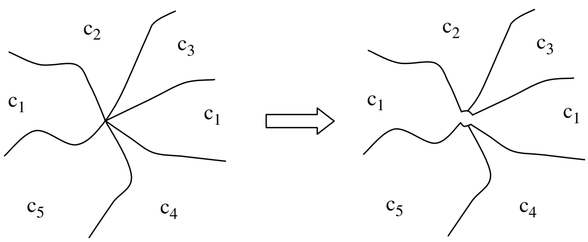

Fig. 24.3

Note that a consequence of (ii) is that we do not need to consider vertices of degree greater than r in an r-colored map. A sequence of theorems now follows, concluding with the main result that *every 5-colored planar map contains a monochrome unit*. Here is an outline of the proof:

1. we show that every 4-colored planar map contains a monochrome unit;
2. we show that every 5-colored planar map containing a vertex of degree 3 contains a monochrome unit;
3. we show that every 5-colored planar map without a monochrome unit must contain a vertex of degree 3;
4. for 2 and 3 both to be true, every 5-colored planar map must contain a monochrome unit.

The Proof: Townsend presents the proof in stages through five theorems.

Theorem 24.2 Let A^0 be a finitely disconnected unit annulus (Fig. 24.1) for which a circle of unit radius contained in its closure, A, has at least one arc of length greater than $\pi/3$ containing no cut points of A. Then any 2-coloring of A^0 contains a monochrome unit.

Outline of Proof The basic argument is as follows (Fig. 24.4).

1. We assume that A^0 is 2-colored and contains no monochrome unit.
2. Points x and y can be selected from A^0, so that they are differently colored and as close together as we want.
3. The points x and y can also be chosen so that (a) x is unit distance from at most one cut point of A, and (b) y is unit distance from no cut points of A.

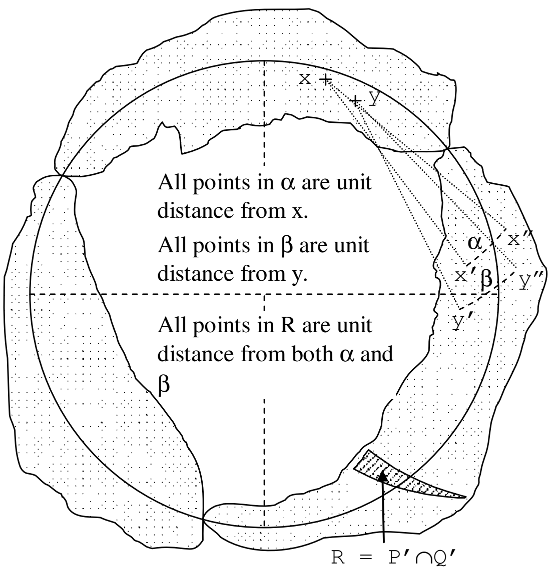

Fig. 24.4

4. Point x subtends an arc α of finite length in A^0, each point of which is unit distance from x, and consequently the opposite color to x. Similarly y subtends an arc β in A^0 which is the opposite color to y.
5. Arc α subtends a two-dimensional region, each point of which is unit distance from a point on α. This region intersects A^0 in a band P' of finite width, each point of which must be the same color as x. A similar region subtended at unit distance by arc β intersects A^0 in a band Q', each point of which is the same color as y.
6. Points x and y can be chosen to lie sufficiently close together to make $R = P' \cap Q'$ non-null.
7. But points in R must simultaneously have the color of x and the color of y, which is impossible. Consequently, the initial two assumptions are incompatible.

The proof hinges on our ability to construct arcs α and β that each does not intersect a cut point of A. This will be true if x is unit distance from at most one cut point of A, and y is unit distance from no cut points of A.

Tool 24.3 Let γ be any simple arc of length L in A^0 with the following properties:

- γ contains at least two points unit distance apart;
- γ contains at most M points, each unit distance from exactly one cut point of A;
- all other points in γ are unit distance from no cut points of A;
- γ is 2-colored with no monochrome units.

Then given $\varepsilon > 0$ there exists an ε-neighbourhood in γ containing a point of each color, one of which is unit distance from no cut points of A and the other of which is unit distance from at most one cut point of A.

Proof Let d(x,y) be the straight line distance between two points x and y on γ, and let δ (x,y) be the distance along γ between x and y.

By assumption there exist two points x_1 and y_1 in γ, not both the same color, with $d(x_1, y_1) = 1$. Let $\varepsilon > 0$ be given. The following algorithm uses the method of bisection to prove the lemma (Fig. 24.5).

1. If $M > 1$ then from the M points in γ that are unit distance from exactly one cut point of A, select the two that are closest together measuring along γ. Let h be the distance between them along γ.
2. If $h < \varepsilon$ then set $\varepsilon = h$.
3. Set $i = 1$.
4. Let w_i be the point in γ mid-way (by arc-length) between x_i and y_i.

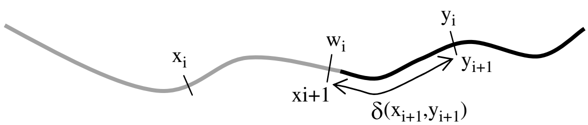

Fig. 24.5

5. If the colors of w_i and x_i are not the same then put $x_{i+1} = x_i$ and $y_{i+1} = w_i$ otherwise put $x_{i+1} = w_i$ and $y_{i+1} = y_i$.
6. If $\delta(x_{i+1}, y_{i+1}) \geq \varepsilon$ increase i by 1 and re-cycle from 4.
7. Points x_{i+1} and y_{i+1} satisfy the requirements.

The algorithm terminates in not more than n cycles, where n is the smallest integer such that $\varepsilon 2^n > L$. ▪

Proof of Theorem 24.2 Let A^0 be 2-colored with no monochrome units. Let N be the number of cut points of A. Let C be a circle of unit radius contained in A. By assumption, C has at least one arc of length greater than $\pi/3$ containing no cut points of A; hence C has an arc containing no cut points of A, whose end points are unit distance apart. There are at most 2N points on C in A^0 that are unit distance from a cut point of A. Some of these may be unit distance from two different cut points of A, but none can be unit distance from more than two cut points of A. By following a path sufficiently close to C it is possible to construct a simple closed curve that, apart form the cut points of A, lies entirely within A^0, which contains at most 2N points in A^0 that are unit distance from a cut point of A, and that contains no points in A^0 that are unit distance from more than one cut point of A. (This curve can merely trace the path of C for the most part, deviating only to bypass any points on C in A^0 that are unit distance from two different cut points of A.) There exists an infinite family Γ of such simple closed curves, for each of which there is an arc of finite length containing two points unit distance apart not separated by a cut point (Fig. 24.6). This must be so since C has two such points, and we can choose the members of Γ to be as close to C as required. For any given $\varepsilon > 0$, this arc contains an ε-neighborhood in which lies a point of each color, one of which is unit distance from at most one cut point of A, and the other of which is unit distance from no cut points of A (by Tool 24.3).

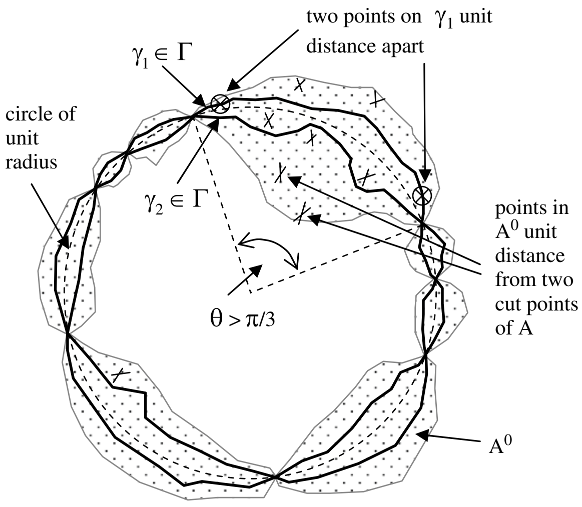

Fig. 24.6

Let γ_1 and γ_2 be members of Γ. Let x and y be two differently colored points in an ε-neighborhood on γ_1 such that x is unit distance from at most one cut point of A and y is unit distance from no cut points of A.

In A^0 there exists an arc α of unit radius and centre x which intersects γ_1 at x' and γ_2 at x'' and no point of which is a cut point of A. (If x is unit distance from one cut point of A then the arc α can be constructed on the other side of x from this cut point.) Arc α cannot be the same color as x, so must be the same color as y. Similarly there exists an arc β in A^0 of unit radius and centre y which intersects γ_1 at y' and γ_2 at y'' and no point of which is a cut point of A. Arc β must be the same color as x.

Let P and Q be sets subtended at unit distance by α and β respectively. P and Q are finitely disconnected unit annuli, each having one cut point at x and y respectively, and each intersecting A^0 in a band of finite width between γ_1 and γ_2 Let these bands be P' and Q' respectively. All points in P' must be the same color as x, and all points in Q' the same color as y. Q' may be considered to be the image of P' under a homeomorphism T which depends on $|x\text{-}y|$. Defining $d(P', Q') = \sup\{|p\text{-}T(p)| : p \in P'\}$ we have $d(P', Q') \to 0$ as $|x\text{-}y| \to 0$; in this sense we say $P' \to Q'$ as $|x\text{-}y| \to 0$. There must then exist $\varepsilon > 0$ such that for $|x\text{-}y| < \varepsilon$, $P' \cap Q' \neq 0$. But all points in $P' \cap Q'$ must simultaneously be colored the same as x and y, which is impossible. Consequently the original assumptions are incompatible, and so if A^0 is 2-colored it must contain a monochrome unit. ∎

Using this result it is possible to exclude two configurations from any 4-coloring of E^2 without monochrome units, and show as a natural consequence that any 4-colored map in E^2 contains a monochrome unit.

Theorem 24.4 Let E^2 be 4-colored. If for some distinct points x and y there exist two simple arcs with endpoints x and y, each, excepting the endpoints, being monochrome but not both the same color, then E^2 contains a monochrome unit.

Proof Let the two simple arcs be γ and δ. If $|x - y| > 1$ then both γ and δ contain a monochrome unit.

Assume $|x - y| \leq 1$. Then the intersection of the sets subtended at unit distance by γ and δ (excluding the endpoints) is a finitely disconnected unit annulus with at most two cut points (Fig. 24.7). This annulus is 2-colored at most, since it cannot contain the colors of γ and δ, and a circle of unit radius contained in its closure has an arc of length greater than $\pi/3$ containing no cut points, and so by Theorem 24.2 the annulus contains a monochrome unit. ∎

Theorem 24.5 If a 4-coloring of E^2 contains two differently colored, bounded, open connected monochrome sets with a common boundary of finite length, then E^2 contains a monochrome unit.

Proof Let G and F be two such sets, and let x and y be two distinct points on the common boundary. Because the closure of G is a simply connected Jordan region, there is a simple arc γ with endpoints x and y which, apart from its endpoints, lies in G. There exists a similar arc δ in F. By theorem 24.4 E^2 contains a monochrome unit. ∎

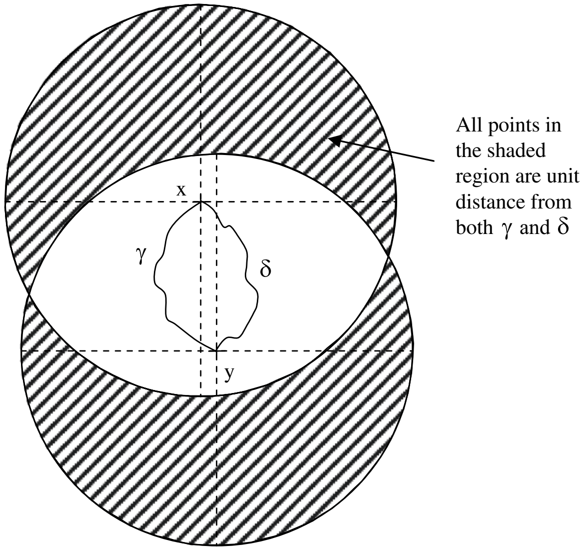

Fig. 24.7

Corollary Every 4-colored planar map contains a monochrome unit.

A similar result involving three sets can be proved for 5-colorings of E^2, and again the consequence is that every 5-colored planar map contains a monochrome unit, but this requires a careful proof.

Theorem 24.6 If a 5-coloring of E^2 contains three disjoint, differently colored, bounded, open, connected, monochrome sets each having a common boundary with each of the other two, and all three having one common boundary point, then E^2 contains a monochrome unit.

Proof Let v be the boundary point common to all three sets and let $a_1,\ a_2$, and a_3 respectively be boundary points common to each pair of sets. We assume these points are distinct and are chosen to be not more than one unit from each other. There are simple closed curves γ_1 colored c_1 containing $v,\ a_1$, and a_2; γ_2 colored c_2 containing $v,\ a_1$, and a_3; and γ_3 colored c_3 containing $v,\ a_2$, and a_3, where in each case the coloring refers to every point on the curve with the possible exception of the points $v,\ a_1,\ a_2$, and a_3 (Fig. 24.8). Let P be the intersection of the sets subtended at unit distance by $\gamma_1,\ \gamma_2$, and γ_3 excepting the points $v,\ a_1,\ a_2$, and a_3. P is either a unit annulus or a finitely disconnected unit annulus with at most three cut points. (A necessary condition for such a cut point to exist is that a set boundary incident to v is an arc of a circle of unit radius; if the cut point exists then it lies at the centre of this circle.) P satisfies the requirements of Theorem 24.2, and since it is 2-colored (viz. not $c_1,\ c_2$ or c_3) it must contain a monochrome unit. ∎

Corollary Every 5-colored planar map containing a vertex of degree 3 contains a monochrome unit.

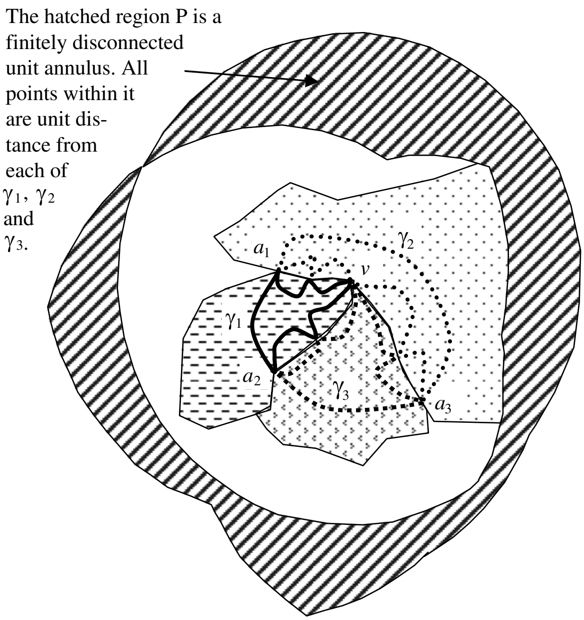

Fig. 24.8

Theorem 24.7 Every 5-colored planar map contains a monochrome unit.

Proof We show (i) that every 5-colored planar map with no monochrome units contains a vertex of degree 3 or 4 and (ii) that every such map containing a vertex of degree 4 also contains a vertex of degree 3.

i. Let v be any vertex in a 5-colored planar map, and assume that this has degree 5. Assume that the map has no monochrome units.
   Let γ be the boundary of one of the regions which has v as a boundary point. Let a be a point on γ that lies on an edge connected to v. Let b be a point on γ that lies on the other edge connected to v (Fig. 24.9). Let c be a point on the edge connected to v that is on the opposite side of va to b. Let d be a point on the edge connected to v that is on the opposite side of vb to a.
   There is a simple closed curve γ_1 passing through v, a, and b all the points of which, except possibly v, a, and b, are colored c_1. There is a simple closed curve γ_2 passing through v, a, and c all the points of which, except possibly v, a, and c, are colored c_2. And there is a simple closed curve γ_3 passing through v, b, and d all the points of which, except possibly v, b, and d, are colored c_3. Let T_2 be the intersection of the sets subtended at unit distance by γ_1 and γ_2 and let T_3 be the intersection of the sets subtended at unit distance by γ_1 and γ_3 (In Fig. 24.9 T_2 is the hatched region and T_3 is the grey region).
   We consider two cases. The first is when the angle θ subtended at v by a line from a to b (through the region enclosed by γ) is greater than π. The interiors

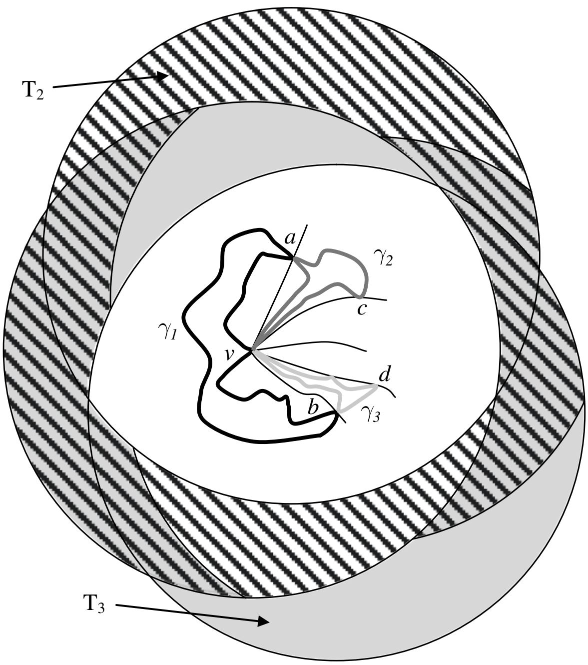

Fig. 24.9

of T_2 and T_3, $T_2{}^0$ and $T_3{}^0$ respectively, are unit annuli with no cut points, and so by Theorem 1 cannot be 2-colored. $T_2{}^0$ must contain regions colored c_3, c_4, and c_5, and $T_3{}^0$ must contain regions colored c_2, c_4, and c_5. The interior of $T_1 = T_2 \cup T_3$ is a 4-colored unit annulus with no cut points.

There is a vertex in $T_1{}^0$. To prove this, assume it is not so. Then there must be edges in $T_1{}^0$ that do not intersect each other in $T_1{}^0$, each of which intersects both the interior and the exterior boundary of T_1. Any such edge, e, must cross both $T_2{}^0$ and $T_3{}^0$. This means that the regions on either side of e must be colored c_4 and c_5. Consequently $T_1{}^0$ is a 2-colored unit annulus, containing no cut points.

The second case is when the angle θ is not greater than π. It is clear, since v is a vertex of degree 5, that the region enclosed by γ may be chosen such that θ is not less than $2\pi/5$. Let a_1 be a point between v and a on the edge on which a lies. Similarly let b_1 be a point between v and b on the edge on which b lies. Choose curve γ_1 so that it passes through a_1 and b_1 as well as v, a, and b and so that all of its points, except possibly v, a, a_1, b_1, and b, are colored c_1. Similarly choose γ_2 to pass through a_1 as well as v, a, and c and γ_3 to pass through b_1 as well as v, b, and d.

Now each of $T_2{}^0$ and $T_3{}^0$ is a finitely disconnected unit annulus with at most one cut point (Fig. 24.10). The single cut point in $T_2{}^0$, say p, only occurs in the event that v, a, and a_1 lie on the circle of unit radius centre p. Similarly the single cut point in $T_3{}^0$, say q, only occurs in the event that v, b, and b_1 lie on the circle

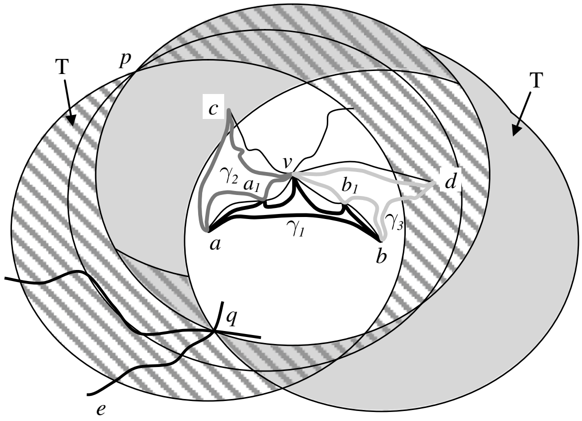

Fig. 24.10

of unit radius centre q. for $T_3{}^0$. The interior of $T_1 = T_2 \cup T_3$ is a 4-colored finitely disconnected unit annulus with at most one cut point. This cut point only occurs in the event that p and q are coincident, and all of v, a, a_1, b_1, and b, lie on the same circle of unit radius. If one of p and q lies on the exterior boundary of T_1 and the other lies on the interior boundary then the length of the arc of the unit circle centre v passing through p and q is θ radians, and this means the distance between p and q is greater than one.

As before we assert there is a vertex in $T_1{}^0$. To prove this, assume it is not so. Then there must be edges in $T_1{}^0$ that do not intersect each other in $T_1{}^0$, each of which intersects both the interior and the exterior boundary of T_1. Any such edge, e, must cross both $T_2{}^0$ and $T_3{}^0$ except in the case that e passes through p and remains entirely within T_3 until it reaches the opposite boundary of T_1, or e passes through q and remains entirely within T_2 until it reaches the opposite boundary of T_1. Note that such an edge e cannot pass through both p and q, since this would imply the existence of a monochrome unit in one of the regions on either side of e. Apart from these exceptional edges every edge in $T_1{}^0$ must separate and regions colored c_4 or c_5. This means that $T_1{}^0$ contains a 2-colored finitely disconnected unit annulus, containing at most two cut points.

Clearly there is a circle of unit radius in T_1 which has an arc of length greater than $\pi/3$ containing no cut points of $T_1{}^0$. Therefore, by Theorem 24.2 $T_1{}^0$ contains a monochrome unit. This is a contradiction of the initial assumption, consequently there must be a vertex in $T_1{}^0$, and since $T_1{}^0$ is 4-colored this vertex is of degree 4 at most.

ii. We show that every 5-colored planar map with no monochrome units containing a vertex of degree 4 also contains a vertex of degree 3.

Suppose v is a vertex of degree 4 in a 5-colored planar map. Let c_1, c_2, c_3, and c_4 be the colors of the four regions of which v is a boundary point. Let *a, b,*

c, and d be points on the four edges incident to v. Let a_1, b_1, c_1, and d_1 be points on the edges between a and v, b and v, c and v, and d and v respectively. Assume that the map has no monochrome units.

There exists a simple closed curve γ_1, defined in the closure of the region colored c1, that passes through v and four of the edge points defined above, and such that every point in γ_1, except possibly v and the four edge points, is colored c1. Similarly, there exist simple closed curves γ_2, γ_3, and γ_4, each of which contains v and four of the edge points, the points on each curve being colored c_2, c_3, and c_4 respectively except possibly v and the edge points. Let the order of the γ_i be chosen such that γ_2 and γ_4 have only the point v in common (Fig. 24.11).

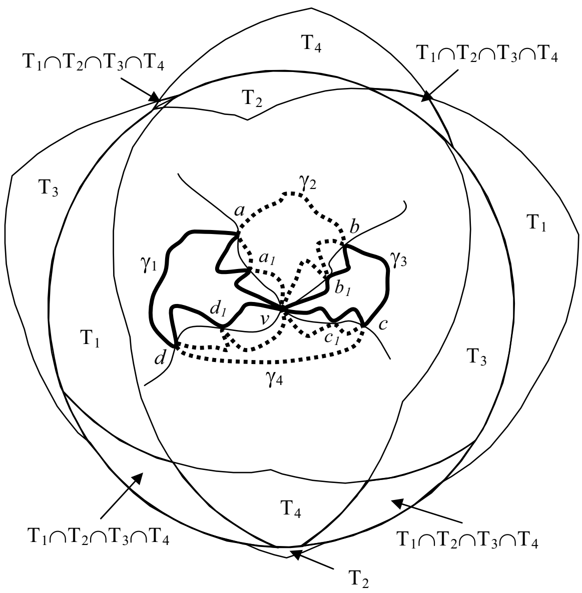

Fig. 24.11

Let T_i, $i = 1, 2, 3, 4$, be the intersection of sets subtended at unit distance by γ_j, $j = 1..4$, $j \neq i$. Set T_i is 2-colored with colors c_i and c_5. Define $T = \cup T_i$. The interior of T, T^0, is a unit annulus with centre v, possibly finitely disconnected with at most two cut points (Fig. 24.11).

Every point within T^0 that is on a boundary of a region of the planar map is a boundary point of at most three regions. Suppose none of these boundary points is a vertex. Then there must exist edges that pass from the interior boundary to the exterior boundary of T which pass through either both of T_1 and T_3 or both of T_2 and T_4. It is possible for an edge to cut T and only cut one of T_1 and T_3 or one of T_2 and T_4, but such an edge must intersect the unit circle centre v at one of four points, these points being cut points (if they exist) of the finitely disconnected annuli which are the interiors of $T_1 \cup T_2$, $T_3 \cup T_4$, $T_1 \cup T_4$, and

$T_2 \cup T_3$. There must be edges crossing T which intersect the circle of unit radius centered on v at points other than these four cut points. (If not then there is an arc of the circle of unit radius centered on v, of length greater than or equal to $\pi/2$ that lies in or on the boundary of a region of the map. But then this region must contain a monochrome unit.) An edge crossing both T_1 and T_3 (or both T_2 and T_4) must separate regions with different colors. But the only color common to both T_1 and T_3 (or both T_2 and T_4) is c_5. We have arrived at a contradiction. Hence, there must be vertices in T^0, and these are of degree 3.

Now, by the corollary of Theorem 24.6 our 5-colored map contains a monochrome unit! ▪